PROBLEM SET 11

JIAHAO HU

Problem 1. *If $X_n \to X$ in probability, then $P_{X_n} \to P_X$ vaguely.*

Proof. Let $F_n(x) = P_{X_n}((-\infty, x]) = P(X_n \leq x)$ and $F(x) = P_X((-\infty, x]) = P(X \leq x)$. Suppose F is continuous at x, then for any $\epsilon > 0$, we can find $\delta > 0$ so that $|y - x| \leq \delta$ implies $|F(y) - F(x)| \leq \epsilon/2$. Since $X_n \to X$ in probability, we have $P(|X_n - X| > \delta) \to 0$ as $n \to \infty$. Therefore, for n big we have $P(|X_n - X| > \delta) \leq \epsilon/2$. So on the one hand, we have

$$\begin{aligned} F_{X_n}(x) = P(X_n \leq x) &\leq P(X_n \leq x, |X_n - X| \leq \delta) + P(|X_n - X| > \delta) \\ &\leq P(X \leq x + \delta) + \epsilon/2 = F(x + \delta) + \epsilon/2 \\ &\leq F(x) + \epsilon. \end{aligned}$$

On the other hand, we also have

$$\begin{aligned} F_X(x) &\leq F_X(x - \delta) + \epsilon/2 = P(X \leq x - \delta) + \epsilon/2 \\ &\leq P(X \leq x - \delta, |X_n - X| \leq \delta) + P(|X_n - X| > \delta) + \epsilon/2 \\ &\leq P(X_n \leq x) + \epsilon = F_{X_n}(x) + \epsilon. \end{aligned}$$

This proves for n big, $|F_n(x) - F(x)| \leq \epsilon$. Thus $F_n(x) \to F(x)$ as $n \to \infty$. Since P_{X_n} are probability distributions, $\|P_{X_n}\| = 1$, therefore by Prop. 7.19, $P_{X_n} \to P_X$ vaguely. $\square$

Problem 2. *Identify $\mathbb{T}^1$ with $\{z \in \mathbb{C} : |z| = 1\}$.*

1. *If $X_1, ..., X_n$ are independent, then $P_{X_1 X_2 \cdots X_n} = P_{X_1} * \cdots * P_{X_n}$.*
2. *If $\{X_j\}$ is a sequence of independent random variables with common distribution λ, the distribution of $\prod_1^n X_j$ converges vaguely to the uniform distribution on $\mathbb{T}^1$ unless X_1 is supported on a finite subset of $\mathbb{T}^1$.*

Proof. (1) Recall that for any two measures on $\mathbb{T}^1$, by definition we have $\mu * \nu(E) = \int\int \chi_E(t_1 \cdot t_2) d\mu(t_1) d\nu(t_2)$ where $t_1 \cdot t_2$ is the group multiplication on $\mathbb{T}^1$ which agrees with the standard multiplication on $\mathbb{C}$.

Let $A(t_1, \dots, t_n) = \prod_1^n t_j$. Then $X_1 \cdots X_n = A(X_1, \dots, X_n)$, so

$$P_{X_1 \cdots X_n} = (P_{(X_1, \dots, X_n)})_A = (\prod_1^n P_{X_j})_A.$$

Therefore,

$$\begin{aligned} P_{X_1 \cdots X_n}(E) = (P_{(X_1, \dots, X_n)})_A &= (\prod_1^n P_{X_j})_A(E) \\ &= \int \cdots \int \chi_E(t_1 \dots t_n) dP_{X_1}(t_1) \cdots dP_{X_n}(t_n) \\ &= P_{X_1} * \cdots P_{X_n}(E). \end{aligned}$$

(2) As we see in HW 9, either λ is supported on a finite subset of $\mathbb{T}^1$ or $|\hat{\lambda}(k)| < 1$ for all $k \neq 0$. In the latter case, by (1) we have

$$|\hat{P}_{X_1\cdots X_n}(k)| = |\prod_1^n \hat{P}_{X_j}(k)| = |\hat{\lambda}(k)|^n \to 0 \quad \text{as } n \to \infty \text{ for } k \neq 0,$$

Notice that the uniform distribution μ on $\mathbb{T}^1$ has,

$$\hat{\mu}(k) = \int_0^1 e^{-2\pi ikx} d\mu(x) = 0 \quad \text{for } k \neq 0.$$

This, together with $\hat{P}_{X_1\cdots X_n}(0) = 1 = \hat{\mu}(0)$, proves $\hat{P}_{X_1\cdots X_n} \to \hat{\mu}$ pointwise. Hence by a similar argument to that in Prop. 8.50, $P_{X_1\cdots X_n} \to \mu$ vaguely.

□

Problem 3. *Given $b \in \mathbb{N} \setminus \{1\}$, let $B = \{0, 1, ..., b-1\}$ and $\Omega = B^{\mathbb{N}}$. Put the discrete topology on B and the product topology on Ω, and let P be the product measure on Ω, where each P_n is b^{-1} times counting measure on B. Let $\{X_n\}_1^\infty$ be the coordinate functions on Ω. Then if $A_1, ..., A_n \subset B$,*

$$\mathbf{Prob}\left(\bigcap_1^n X_j^{-1}(A_j)\right) = b^{-n} \prod_1^n |A_j|$$

and $P(\{\omega\}) = 0$ for all $\omega \in \Omega$.

Proof. Since $\bigcap_1^n \left(X_j^{-1}(A_j)\right) = \prod_1^n A_j \times \ B^{\mathbb{N}-\{1,\dots,n\}}$, by definition of product measure we have,

$$\mathbf{Prob}\left(\bigcap_1^n X_j^{-1}(A_j)\right) = \prod_1^n P_i(A_i) \times 1 = b^{-n} \prod_1^n |A_j|.$$

Then since $\omega \in \cap_1^n X_j^{-1}(\{X_j(\omega)\})$ for all n, it follows

$$P(\{\omega\}) \leq b^{-n} \prod_1^n 1 = b^{-n} \to 0 \quad \text{as } n \to \infty.$$

□

Problem 4. *Prove the following. Let*

$$\Omega' = \{\omega \in \Omega : X_n(\omega) \neq 0 \textit{ for infinitely many } n\}.$$

Then $\Omega \setminus \Omega'$ is countable and $P(\Omega') = 1$. Define $F : \Omega \to [0,1]$ by $F(\omega) = \sum_1^\infty X_n(\omega) b^{-n}$. Then $F|_{\Omega'}$ is a bijection from Ω' to $(0,1]$ which maps $\mathcal{B}_{\Omega'}$ bijectively onto $\mathcal{B}_{(0,1]}$.

Proof. It is clear $\omega \mapsto \sum_1^\infty X_j(\omega) x^j$ defines a bijection between Ω and all power series with coefficients in B. Under this bijection, $\Omega \backslash \Omega'$ corresponds to polynomials with coefficients in finite set B, which is clearly countable (by considering degree). Therefore $\Omega \backslash \Omega'$ is countable as well. And since a single point has measure zero in Ω, so does the countable set $\Omega \backslash \Omega'$, hence $P(\Omega') = P(\Omega) = 1$.

Notice that $\mathcal{B}_{\Omega'}$ is generated by sets $\{\omega \in \Omega' : X_j(\omega) = a_j\}$, and hence generated by $\{\omega \in \Omega' : X_1(\omega) = a_1, \dots, X_n(\omega) = a_n\}$. Under F, the set $\{\omega \in \Omega' : X_1(\omega) = a_1, \dots, X_n(\omega) = a_n\}$ is mapped to all numbers of the form $0.a_1 a_2 \dots a_n \dots$ under base b decimal expansion. This is precisely the interval $(\frac{j}{n^n}, \frac{j+1}{b^n}]$ where $j = a_1 b^{n-1} + \cdots + a_n$. Now since intervals of the form $(\frac{j}{b^n}, \frac{j+1}{b^n}]$ generate $\mathcal{B}_{(0,1]}$,

F maps $\mathcal{B}_{\Omega'}$ bijectively onto $\mathcal{B}_{(0,1]}$. Moreover, from Problem 3, the set $\{\omega \in \Omega' : X_1(\omega) = a_1, \ldots, X_n(\omega) = a_n\}$ has probability b^{-n} which agrees with the Lebesgue measure of $(\frac{j}{b^n}, \frac{j+1}{b^n}]$, this proves F carries P to the Lebesgue measure. □

Problem 5. *(Borel's normal number theorem) A number $x \in (0,1]$ is called* normal *in base b if the digits $0, 1, ..., b-1$ occur with equal frequency in its base b decimal expansion, that is, if $n^{-1} card\{m \in \{1, \ldots, n\} : X_m(F^{-1}(x)) = j\} \to b^{-1}$ as $n \to \infty$ for $j = 0, 1, \ldots, b-1$. Almost every $x \in (0,1]$ (with respect to Lebesgue measure) is normal in base b for every b.*

Proof. Let $Y_{m,j}(\omega) = 1$ if $X_m(\omega) = j$ and $Y_{m,j}(\omega) = 0$ otherwise. Since X_m's are i.i.d., so are $Y_{m,j}$'s with $E(Y_{m,j}) = b^{-1}$ (by an easy calculation). Since Y_m are bounded on finite measure space, $Y_m \in L^1$, the strong law of large numbers implies $\frac{1}{n}\sum_1^n Y_{m,j} \to b^{-1}$ almost surely. Notice that

$$\text{card}\{m \in \{1, \ldots, n\} : X_m(\omega) = j\} = \sum_1^n Y_m(\omega)$$

and F bijectively takes P to Lebesgue measure on $(0,1]$, we conclude

$$\lim_{n\to\infty} n^{-1}\text{card}\{m \in \{1, \ldots, n\} : X_m(F^{-1}(x)) = j\} = b^{-1}$$

for almost every $x \in (0,1]$.

Now we may finish the proof inductively as follow. First there is a probability 1 subspace $\Omega'_0 \subset \Omega'$ on which $\frac{1}{n}\sum_1^n Y_{m,0}$ converges to b^{-1}. Then we repeat the argument for Ω'_0 to get a probability 1 subspace $\Omega'_{0,1} \subset \Omega'_0$ on which $\frac{1}{n}\sum_1^n Y_{m,1}$ converges to b^{-1}. Keep doing this, we eventually get a probability 1 subspace $\Omega'_{0,1,\ldots,b-1}$ on which $\frac{1}{n}\sum_1^n Y_{m,j}$ converges to b^{-1} for all $j = 0, 1, \ldots, b-1$. Notice $F(\Omega'_{0,1,\ldots,b-1})$ is a subset of normal numbers, so almost every $x \in (0,1]$ is normal. □